Algebra II — Lesson 0430

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1. Introduction

We begin our discussion of probability with a simple example. We all are familiar with dice. A die is a cube with differing numbers of dots on each side, one through six. We roll a die, and one side is up; this is called an *experiment*. We count the dots on that side, and call this the *outcome* of the experiment. The set of all possible outcomes is called the *sample space*.

These outcomes are modeled by the numbers one through six. Thus, in this case, the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

What is the probability of rolling a two? Well, all of the numbers one through six have an equal likelihood of being the outcome, and two is just one of these six values, so we say that the chances of rolling a two is one in six.

What is the probability of rolling a number two or less? An *event* is a subset of the sample space. In this case, the event we are looking for is the set $E = \{1, 2\}$.

There are two out outcomes in the event E, and six possible outcomes is the sample space S, so the probability of this event is two in six, or one in three. We write $P(E) = \frac{1}{3}$.

The *cardinality* of a set is the number of elements in it. The cardinality of set A is denoted $|A|$. In our example, $|S| = 6$ and $|E| = 2$.

We now make a formal definition.

2. Definition of Probability

Definition 1. A *uniform probability space* consists of a set S, called the *sample space*, all of its subsets, which are called *events*, and a function P whose domain is the set of events, called the *probability measure*. If $E \subset S$, then the probability of E is defined as

$$P(E) = \frac{|E|}{|S|}.$$

Since $E \subset S$, then $|E| \leq |S|$. Therefore, $P(E)$ is a real number between zero and one.

3. Simple Examples

Example 1. You roll one die. Find the probability of rolling an even number.

Solution. An experiment is rolling one die. An outcome is an integer between one and six. Thuse, the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Note that $|S| = 6$.

Let E be the event of rolling an even number. Then $E = \{2, 4, 6\}$, and $|E| = 3$. Thus

$$P(E) = \frac{|E|}{|S|} = \frac{3}{6} = 0.5.$$

□

Example 2. We roll two dice and add there values. Which number is the most likely sum? Find that number, and its probability.

Solution. We have two dice, the first and the second. If we roll them both, we get two numbers; say the first is a two and the second is a four. The mathematical object that best models this is an ordered pair of integers between one and six, such as $(2, 4)$. That is, the outcomes are ordered pairs, and the set of all possible rolls is the sample space:

$$\begin{aligned} S = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}. \end{aligned}$$

Note that $|S| = 6 \times 6 = 36$.

By adding the values, it is possible to get a number between two and twelve. The events are the sets of ordered pairs whose entries add up to a given sum:

- $E_2 = \{(1,1)\}$ $\quad |E_2| = 1 \quad P(E_2) = \frac{1}{36} = 0.0278$
- $E_3 = \{(1,2),(2,1)\}$ $\quad |E_3| = 2 \quad P(E_3) = \frac{2}{36} = 0.0556$
- $E_4 = \{(1,3),(2,2),(3,1)\}$ $\quad |E_4| = 3 \quad P(E_4) = \frac{3}{36} = 0.0833$
- $E_5 = \{(1,4),(2,3),(3,2),(4,1)\}$ $\quad |E_5| = 4 \quad P(E_5) = \frac{4}{36} = 0.1111$
- $E_6 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ $\quad |E_6| = 5 \quad P(E_6) = \frac{5}{36} = 0.1389$
- $E_7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ $\quad |E_7| = 6 \quad P(E_7) = \frac{6}{36} = 0.1667$
- $E_8 = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$ $\quad |E_8| = 5 \quad P(E_8) = \frac{5}{36} = 0.1389$
- $E_9 = \{(3,6),(4,5),(5,4),(6,3)\}$ $\quad |E_9| = 4 \quad P(E_9) = \frac{4}{36} = 0.1111$
- $E_{10} = \{(4,6),(5,5),(6,4)\}$ $\quad |E_{10}| = 3 \quad P(E_{10}) = \frac{3}{36} = 0.0833$
- $E_{11} = \{(5,6),(6,5)\}$ $\quad |E_{11}| = 2 \quad P(E_{11}) = \frac{2}{36} = 0.0556$
- $E_{12} = \{(6,6)\}$ $\quad |E_{12}| = 1 \quad P(E_{12}) = \frac{1}{36} = 0.0278$

The most likely number to roll is seven. □